Propagation of Light Through Atmospheric Turbulence

Lecture 5, ASTR 289

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Outline of today's lecture

1. Review: Use phase structure function $D_\phi \sim r^{2/3}$ to calculate statistical properties of light propagation thru index of refraction variations

2. Use these statistics to derive the atmospheric coherence length, r_0

3. Use r_0 to calculate key quantities for AO performance

Review: Kolmogorov Turbulence

- 1-D power spectrum of velocity fluctuations: $k = 2\pi / \lambda$

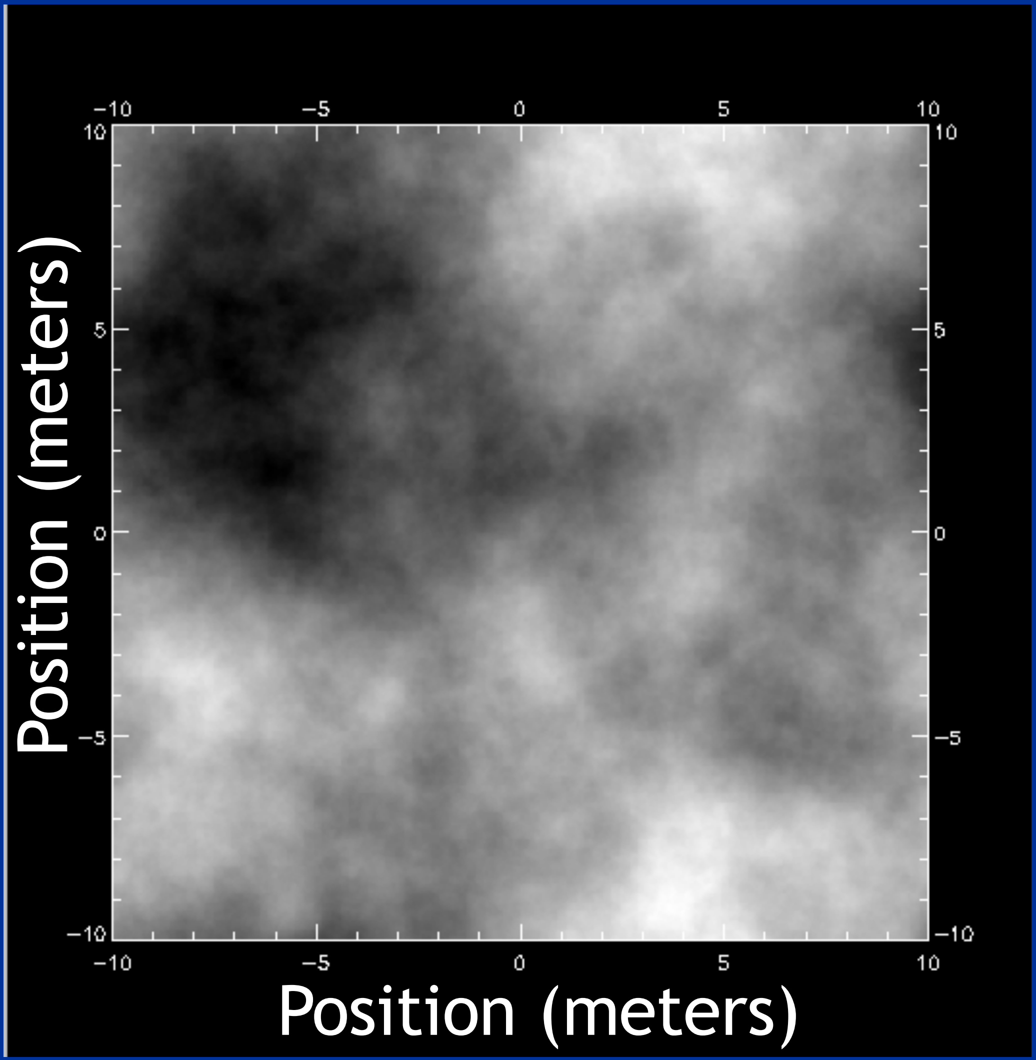
  $\Phi(k) \sim k^{-5/3}$ (one dimension)

- 3-D power spectrum: $\Phi^{3D}(k) \sim \Phi / k^2$

  $\Phi^{3D}(k) \sim k^{-11/3}$ (3 dimensions)

- Valid for fully developed turbulence, over the "inertial range" between the outer scale L_0 and the inner scale l_0

What does a Kolmogorov distribution of phase look like?



- A Kolmogorov "phase screen" courtesy of Don Gavel
- Shading (black to white) represents phase differences of ~1.5 μm
- r_0 = 0.4 meter

Structure function for atmospheric fluctuations, Kolmogorov turbulence

- Structure functions for temperature and index of refraction

$$D_T(r) = \left\langle \left[T(x) - T(x+r) \right]^2 \right\rangle = C_T^2 r^{2/3}$$

$$D_N(r) = \left\langle \left[N(x) - N(x+r) \right]^2 \right\rangle = C_N^2 r^{2/3}$$

- For atmospheric turbulence, C_N^2 and C_T^2 are functions of altitude *z*: $C_N^2(z)$ and $C_T^2(z)$

Spatial Coherence Function

- Spatial coherence function of a field is defined as

$$B_h(\vec{r}) \equiv \left\langle \Psi(\vec{x})\Psi^*(\vec{x}+\vec{r}) \right\rangle$$

Covariance for complex fn's

  - $B_h(\vec{r})$ is a measure of how "related" the field Ψ is at one position (e.g. x) to its values at neighboring positions ($x + r$).

Since $\Psi(\vec{x}) = \exp[i\phi(\vec{x})]$ and $\Psi^*(\vec{x}) = \exp[-i\phi(\vec{x})]$,

$$B_h(\vec{r}) = \left\langle \exp i[\phi(\vec{x}) - \phi(\vec{x}+\vec{r})] \right\rangle$$

Result of long computation of the spatial coherence function B_h (r)

$$B_h(\vec{r}) = \exp\left[-D_\phi(\vec{r})/2\right] = \exp\left[-\frac{1}{2}\left(2.914\, k^2 r^{5/3} \int_0^\infty dh\, C_N^2(h)\right)\right]$$

For a slant path insert multiplicative factor *(sec θ)*$^{5/3}$ to account for dependence on zenith angle *θ*

Digression to define optical transfer function (OTF)

- Imaging in the presence of imperfect optics (or aberrations in atmosphere): in intensity units

Image = Object $\otimes$ Point Spread Function

convolved with

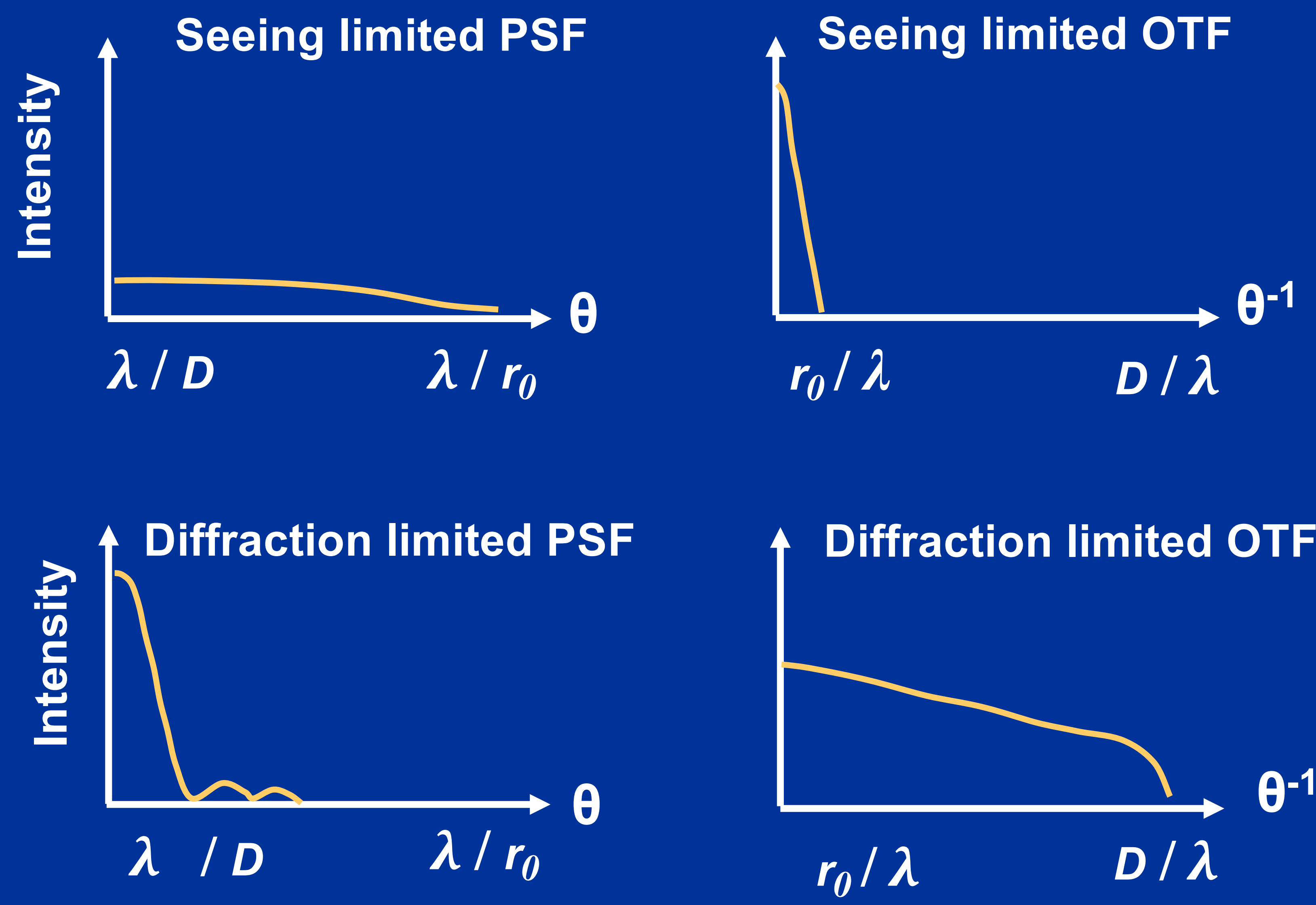
$$I = O \otimes PSF \equiv \int d\vec{x}\; O(\vec{x} - \vec{r})\; PSF\,(\vec{x})$$

- Take Fourier Transform: $\tilde{F}(I) = \tilde{F}(O)\; \tilde{F}(PSF)$

- Optical Transfer Function = Fourier Transform of PSF

$$\tilde{F}(I) = \text{OTF} \times \tilde{F}(O)$$

Examples of PSF's and their Optical Transfer Functions



Derive the atmospheric coherence length, r_0

- Define r_0 as the telescope diameter where optical transfer functions of telescope and atmosphere are equal:

$$OTF_{telescope} = OTF_{atmosphere}$$

- We will then be able to use r_0 to derive relevant timescales of turbulence, and to derive "Isoplanatic Angle":
  - Describes how AO performance degrades as astronomical targets get farther from guide star

First need optical transfer function of the telescope in the presence of turbulence

- OTF for the whole imaging system (telescope plus atmosphere)

$$S(f) = B(f)\, T(f)$$

Here *B (f)* is the optical transfer fn. of the atmosphere and *T (f)* is the optical transfer fn. of the telescope (units of *f* are cycles per meter).

f is often normalized to cycles per diffraction-limit angle (λ / *D*).

- Measure resolving power $\Re$ of the imaging system by

$$\Re = \int df\; S(f) = \int df\; B(f)\, T(f)$$

Derivation of r_0

- $\Re$ of a perfect telescope with a purely circular aperture of (small) diameter *d* is

$$\Re = \int df \; T(f) = \frac{\pi}{4}\left(\frac{d}{\lambda}\right)^2$$

(uses solution for diffraction from a circular aperture)

- Define a circular aperture $d = r_0$ such that the $\Re$ of the telescope (without any turbulence) is equal to the $\Re$ of the atmosphere alone:

$$\int df \; B(f) = \int df \; T(f) \equiv \frac{\pi}{4}\left(\frac{r_0}{\lambda}\right)^2$$

Derivation of r_0, continued

- Now we have to evaluate the contribution of the atmosphere's OTF: $\int df \; B(f)$

- $B(f) = B_h(\lambda f)$ (to go from units of cycles per meter to cycles per wavelength) – see slide 8

$$B_h(\vec{r}) = \exp\left[-\frac{1}{2}\left(2.914\, k^2 r^{5/3} \int_0^\infty dh C_N^2(h)\right)\right]$$

$$\text{Also,} \quad B(f) = B_h(\lambda f) = \exp\left(-K\, f^{5/3}\right)$$

(Kolmogorov)

Derivation of r_0, continued

- Now we need to do the integral in order to solve for r_0 :

$$\frac{\pi}{4}\left(\frac{r_0}{\lambda}\right)^2 = \int df\ B(f) = \int df \exp\left(-K\ f^{5/3}\right) = \frac{6\pi}{5}\ \Gamma\left(\frac{6}{5}\right) K^{-6/5}$$

- Solve for *K*: $K = 3.44\left(\frac{r_0}{\lambda}\right)^{-5/3}$

$$B(f) = \exp\left[-3.44\left(\frac{\lambda f}{r_0}\right)^{5/3}\right] = \exp\left[-3.44\left(\frac{r}{r_0}\right)^{5/3}\right]$$

OTF of atmosphere

Replace λf by r

Derivation of r_0, concluded

$$3.44\left(\frac{r}{r_0}\right)^{5/3} = \frac{1}{2}\left[2.914 k^2 r^{5/3} \sec\varsigma \int dh C_N^2(h)\right]$$

$$r_0 = \left[0.423 k^2 \sec\varsigma \int dh C_N^2(h)\right]^{-3/5}$$

Hooray!

Scaling of r_0

- We will show that r_0 sets scale of all AO correction

$$r_0 = \left[0.423 k^2 \sec\varsigma \int_0^H C_N^2(z)\, dz \right]^{-3/5} \propto \lambda^{6/5} \left(\sec\varsigma\right)^{-3/5} \left[\int C_N^2(z)\, dz \right]^{-3/5}$$

- r_0 gets smaller when turbulence is strong (C_N^2 large)
- r_0 gets bigger at longer wavelengths: AO is easier in the IR than with visible light
- r_0 gets smaller quickly as telescope looks toward the horizon (larger zenith angles ζ)

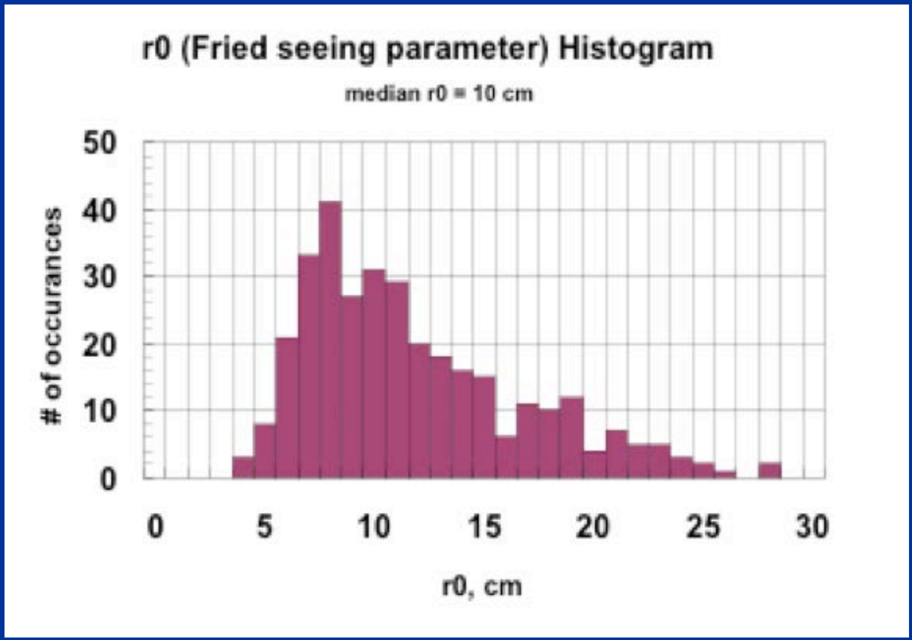
Typical values of r_0

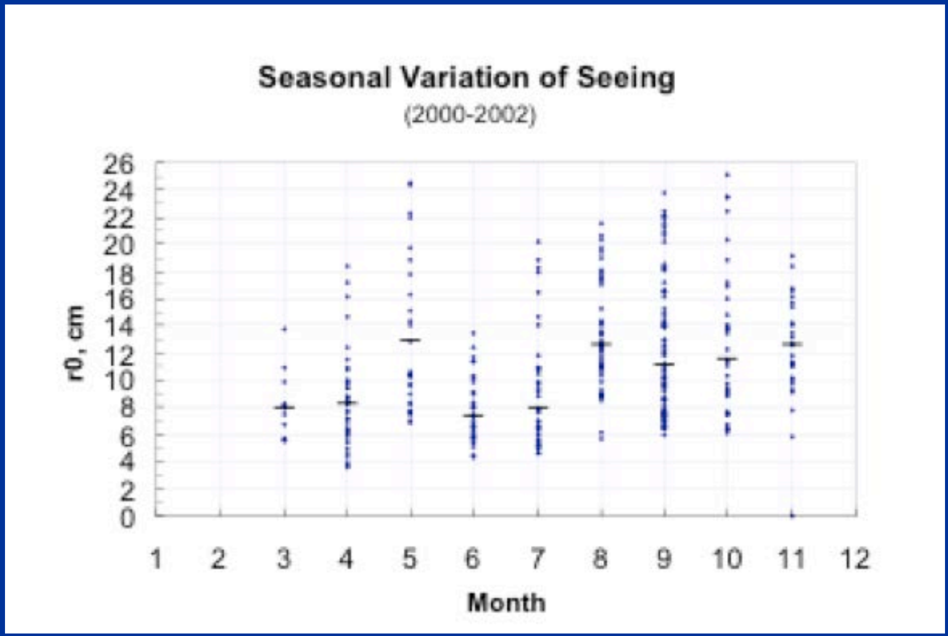
- Usually r_0 is given at a 0.5 micron wavelength for reference purposes.
- It's up to you to scale it by $\lambda^{6/5}$ to evaluate r_0 at your favorite wavelength.
- At excellent sites such as Mauna Kea in Hawaii, r_0 at $\lambda = 0.5$ micron is 10 - 30 cm.
- But there is a big range from night to night, and at times also within a night.

Several equivalent meanings for r_0

- Define r_0 as telescope diameter where optical transfer functions of the telescope and atmosphere are equal (the calculation we just did)
- r_0 is separation on the telescope primary mirror where phase correlation has fallen by 1/e
- $(D/r_0)^2$ is approximate number of speckles in short-exposure image of a point source
- D/r_0 sets the required number of degrees of freedom of an AO system
- Can you think of others?

Seeing statistics at Lick Observatory (Don Gavel and Elinor Gates)





- Left: Typical shape for histogram has peak toward lower values of r_0 with long tail toward large values of r_0
- Huge variability of r_0 within a given night, week, or month
- Need to design AO systems to deal with a significant range in r_0

Effects of turbulence depend on size of telescope, through D/r_0

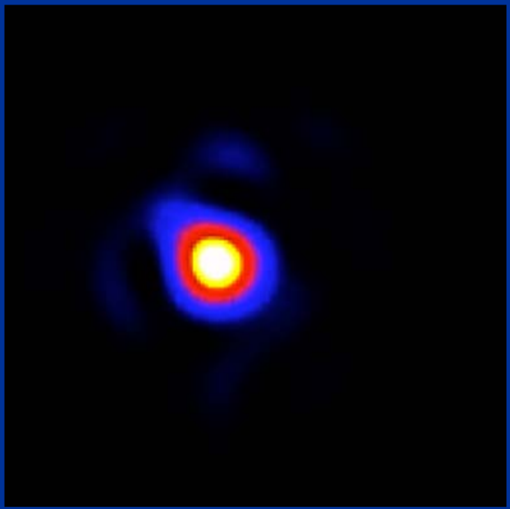
- For telescope diameter $D > (2 - 3) \times r_0$:

  Dominant effect is "image wander"

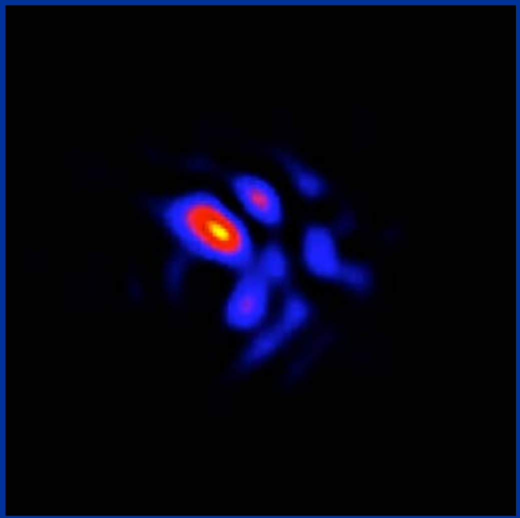
- As D becomes $\gg r_0$:

  Many small "speckles" develop

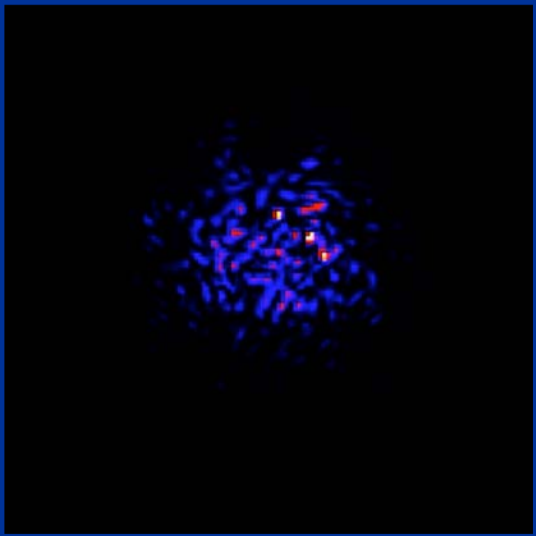
- Computer simulations by Nick Kaiser: image of a star, r_0 = 40 cm



D = 1 m

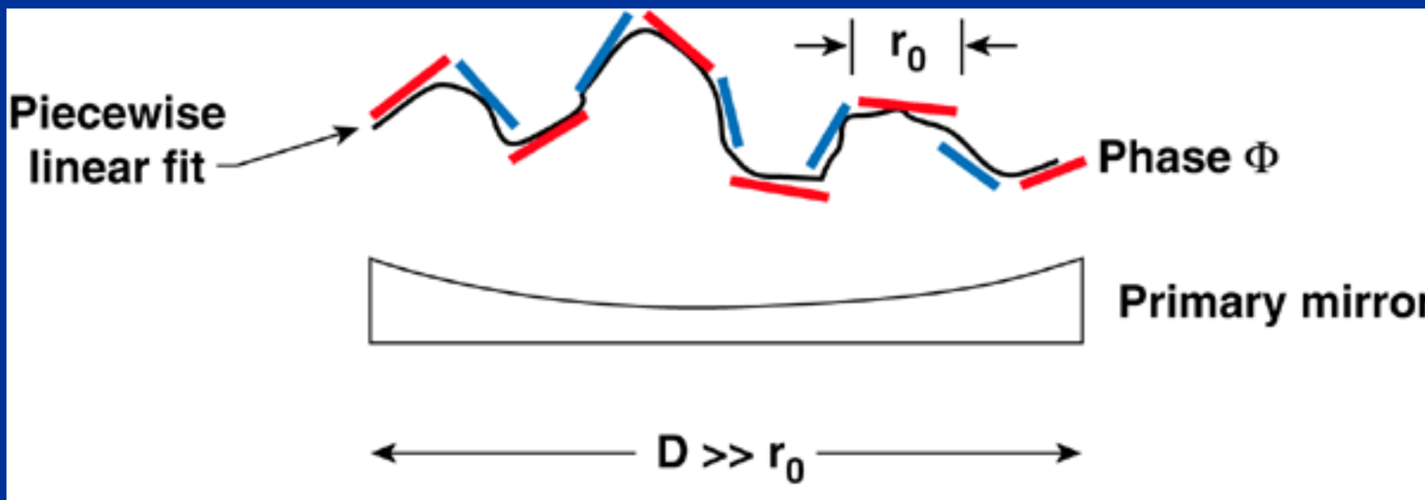


D = 2 m



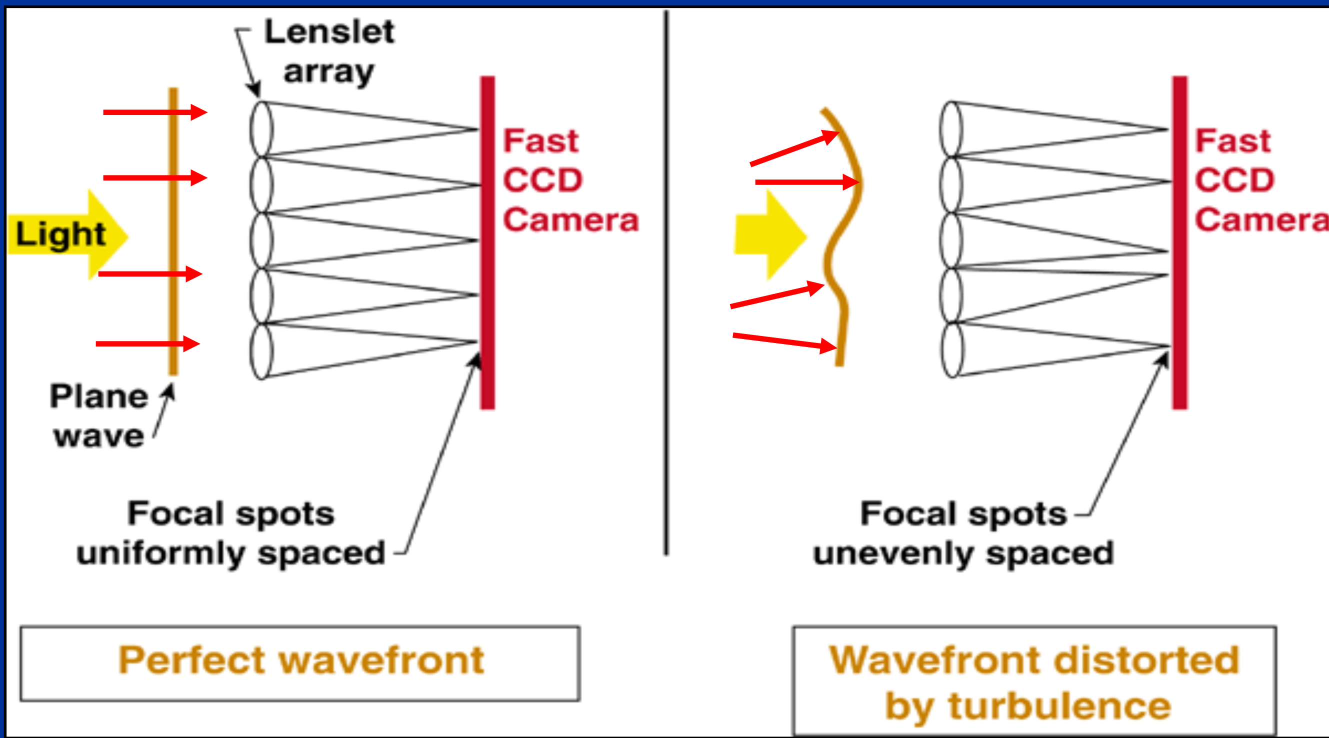
D = 8 m

Implications of r_0 for AO system design: 1) Deformable mirror complexity



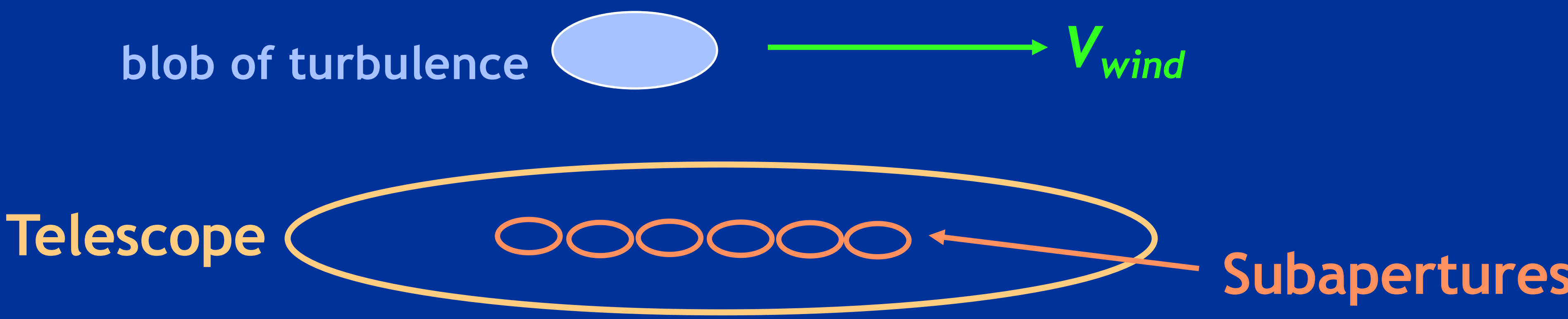
- Need to have DM actuator spacing ~ r_0 in order to fit the wavefront well
- Number of "subapertures" or actuators needed is proportional to area ~ $(D / r_0)^2$

Implications of r_0 for AO system design: 2) Wavefront sensor and guide star flux



- Diameter of lenslet $\leq r_0$
  - Need wavefront measurement at least for every subaperture on deformable mirror
- Smaller lenslets need brighter guide stars to reach same signal to noise ratio for wavefront measurement

Implications of r_0 for AO system design: 3) Speed of AO system



- Timescale over which turbulence within a subaperture changes is

$$\tau \sim \frac{subaperture\ diameter}{V_{wind}} \sim \frac{r_0}{V_{wind}}$$

- Smaller r_0 (worse turbulence) ⇒ need faster AO system
- Shorter WFS integration time ⇒ need brighter guide star

Summary of sensitivity to r_0

- For smaller r_0 (worse turbulence) need:
  - Smaller sub-apertures
    - » More actuators on deformable mirror
    - » More lenslets on wavefront sensor
  - Faster AO system
    - » Faster computer, lower-noise wavefront sensor detector because each frame you read brings noise
  - Much brighter guide star (natural star or laser)

Interesting implications of r_0 scaling for telescope scheduling

- If AO system must work under (almost) all atmospheric conditions, will be quite expensive
  - Difficulty and expense scale as a high power of r_0
  - Need more and more actuators for smaller values of r_0

- Two approaches:
  - Spend the extra money on AO in order to be able to use almost all the observing time allocated to AO
  - Use flexible schedule algorithm that only turns AO system "on" when r_0 is larger than a particular value (turbulence is weaker than a particular value)

Next: All sorts of good things that come from knowing r_0

- Timescales of turbulence

- Isoplanatic angle: AO performance degrades as astronomical targets get farther from guide star

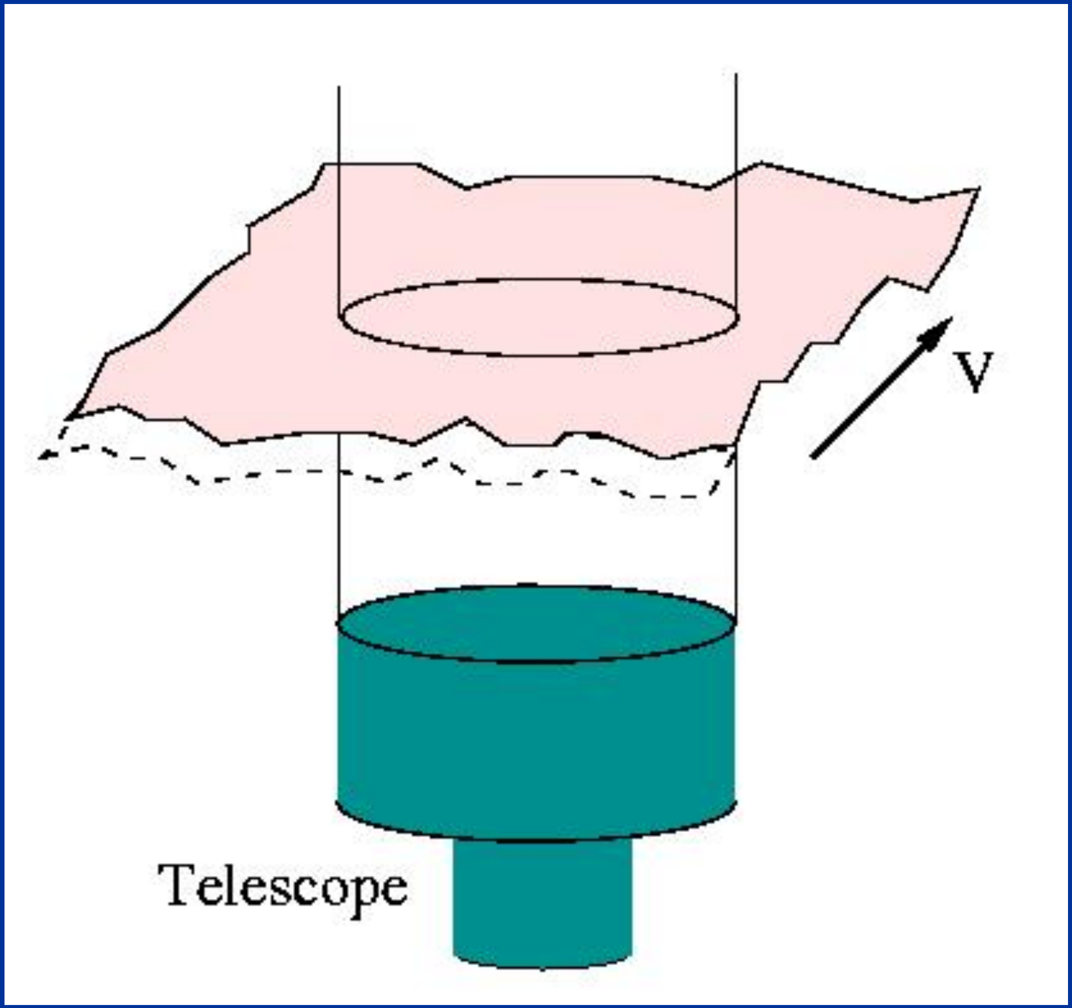
A simplifying hypothesis about time behavior

- Almost all work in this field uses "Taylor's Frozen Flow Hypothesis"
  - Entire spatial pattern of a random turbulent field is transported along with the wind
  - Turbulent eddies do not change significantly as they are carried across the telescope by the wind
  - True if typical velocities within the turbulence are small compared with the overall fluid (wind) velocity

- Allows you to infer time behavior from measured spatial behavior and wind speed:

$$\frac{\partial \vec{u}}{\partial t} = -\vec{u} \bullet \nabla \vec{u}$$

Cartoon of Taylor Frozen Flow



- From Tokovinin tutorial at CTIO:
- http://www.ctio.noao.edu/~atokovin/tutorial/

What is typical timescale for flow across distance r_0 ?

- Time for wind to carry frozen turbulence over a subaperture of size r_0 (Taylor's frozen flow hypothesis):

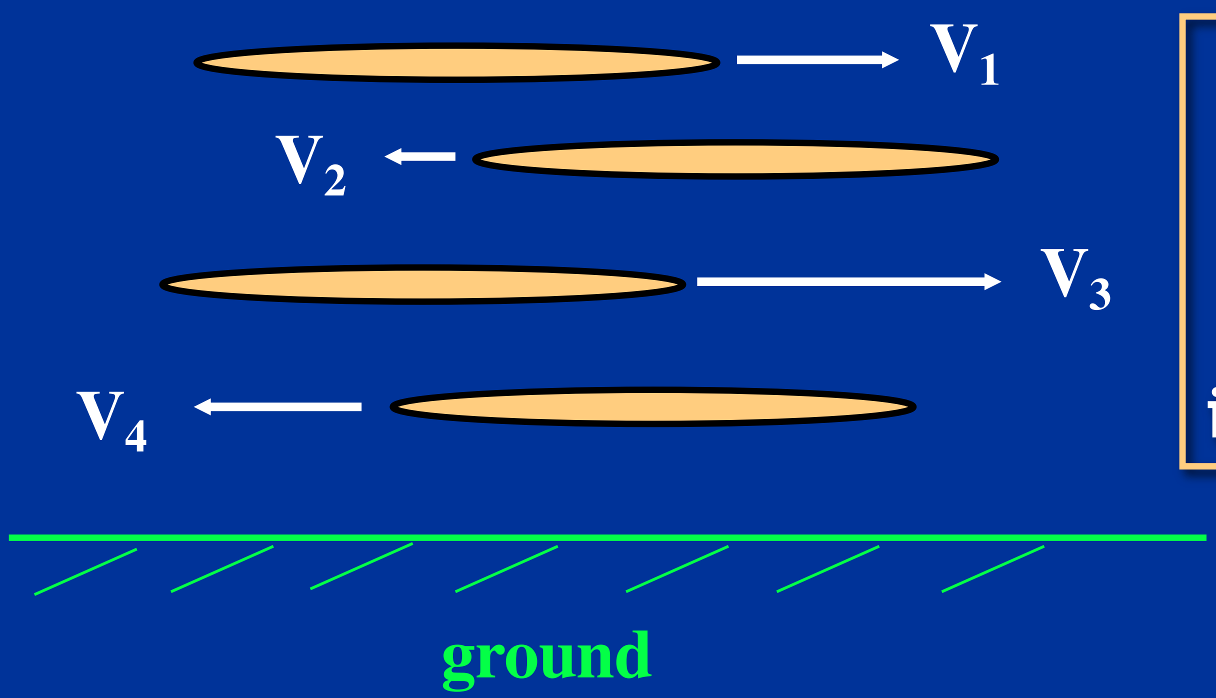
$$\tau_0 \sim r_0 / V$$

- Typical values at a good site, for V = 20 m/sec:

Wavelength (μm)	r_0	$\tau_0 = r_0 / V$	$f_0 = 1/\tau_0 = V / r_0$
0.5	10 cm	5 msec	200 Hz
2	53 cm	27 msec	37 Hz
10	3.6 meters	180 msec	5.6 Hz

But what wind speed should we use?

- If there are layers of turbulence, each layer can move with a different wind speed in a different direction!
- And each layer has different C_N^2



Concept Question:
What would be a plausible way to weight the velocities in the different layers?

Rigorous expressions for τ_0 take into account different layers

- $f_G \equiv$ Greenwood frequency $\equiv 1 / \tau_0$

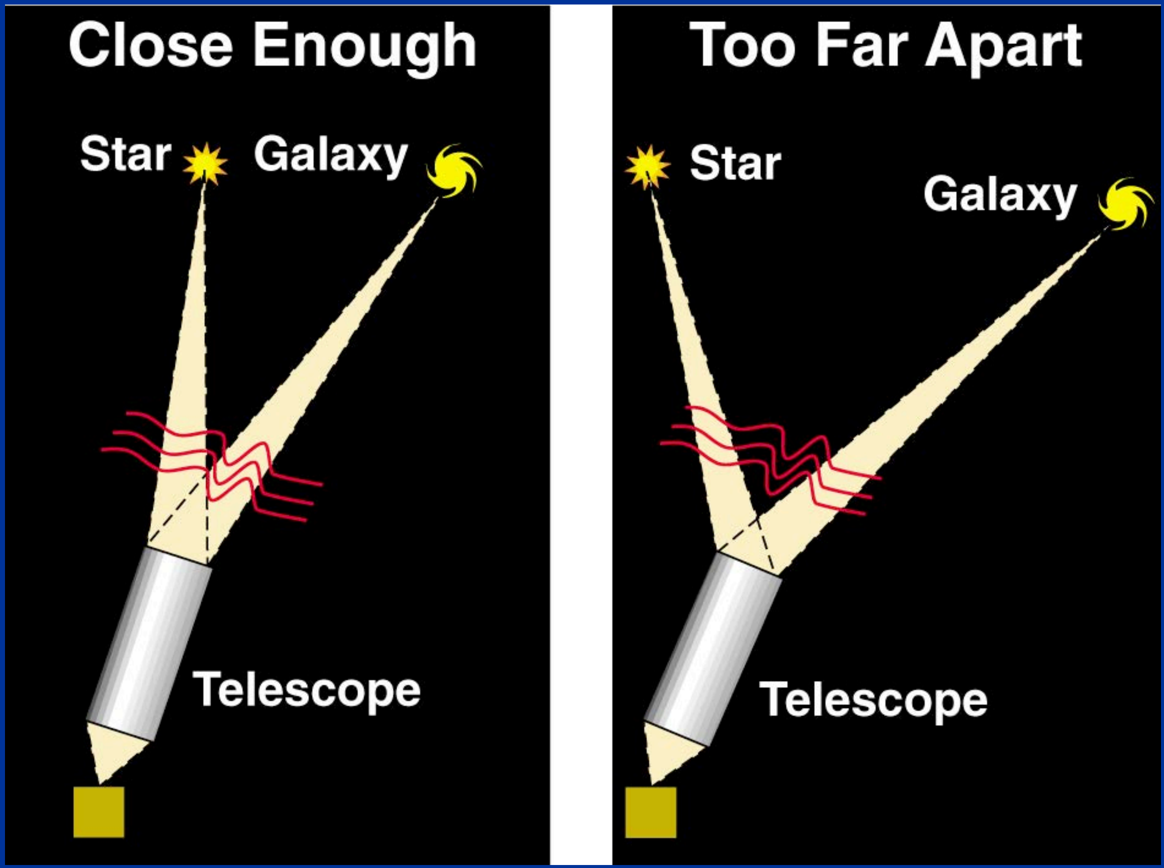
$$\tau_0 \sim 0.3 \left(\frac{r_0}{\bar{V}} \right) \text{ where } \bar{V} \equiv \left[\frac{\int dz\, C_N^2(z)\, |V(z)|^{5/3}}{\int dz\, C_N^2(z)} \right]^{3/5}$$

$$\tau_0 = f_G^{-1} = \left[0.102\, k^2 \sec\zeta \int_0^\infty dz\, C_N^2(z)\, |V(z)|^{5/3} \right]^{-3/5} \propto \lambda^{6/5}$$

What counts most are high velocities V where C_N^2 is big

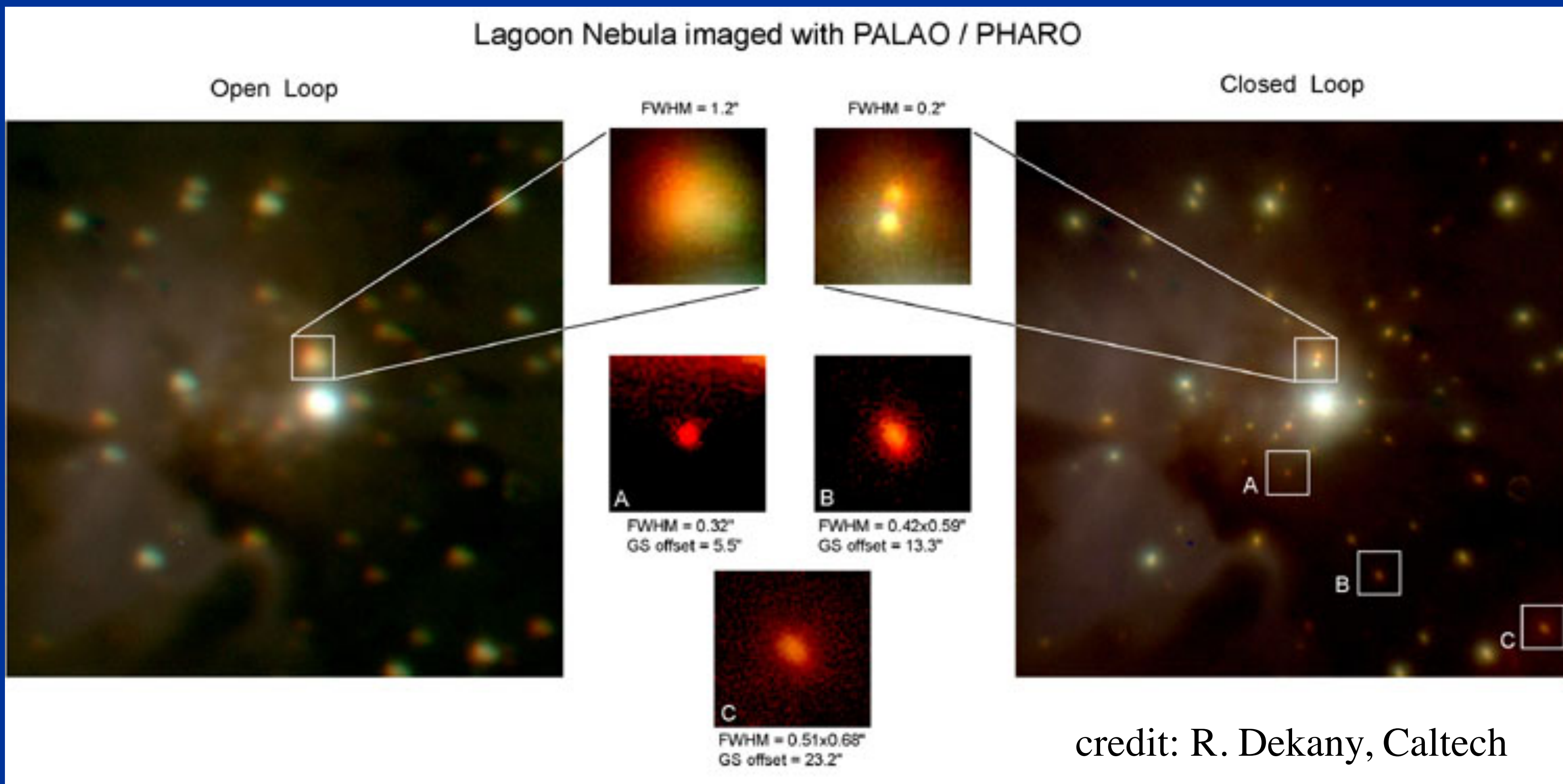
Isoplanatic Angle: angle over which turbulence is still well correlated

Same turbulence



Different turbulence

Anisoplanatism: Nice example from first Palomar AO system



credit: R. Dekany, Caltech

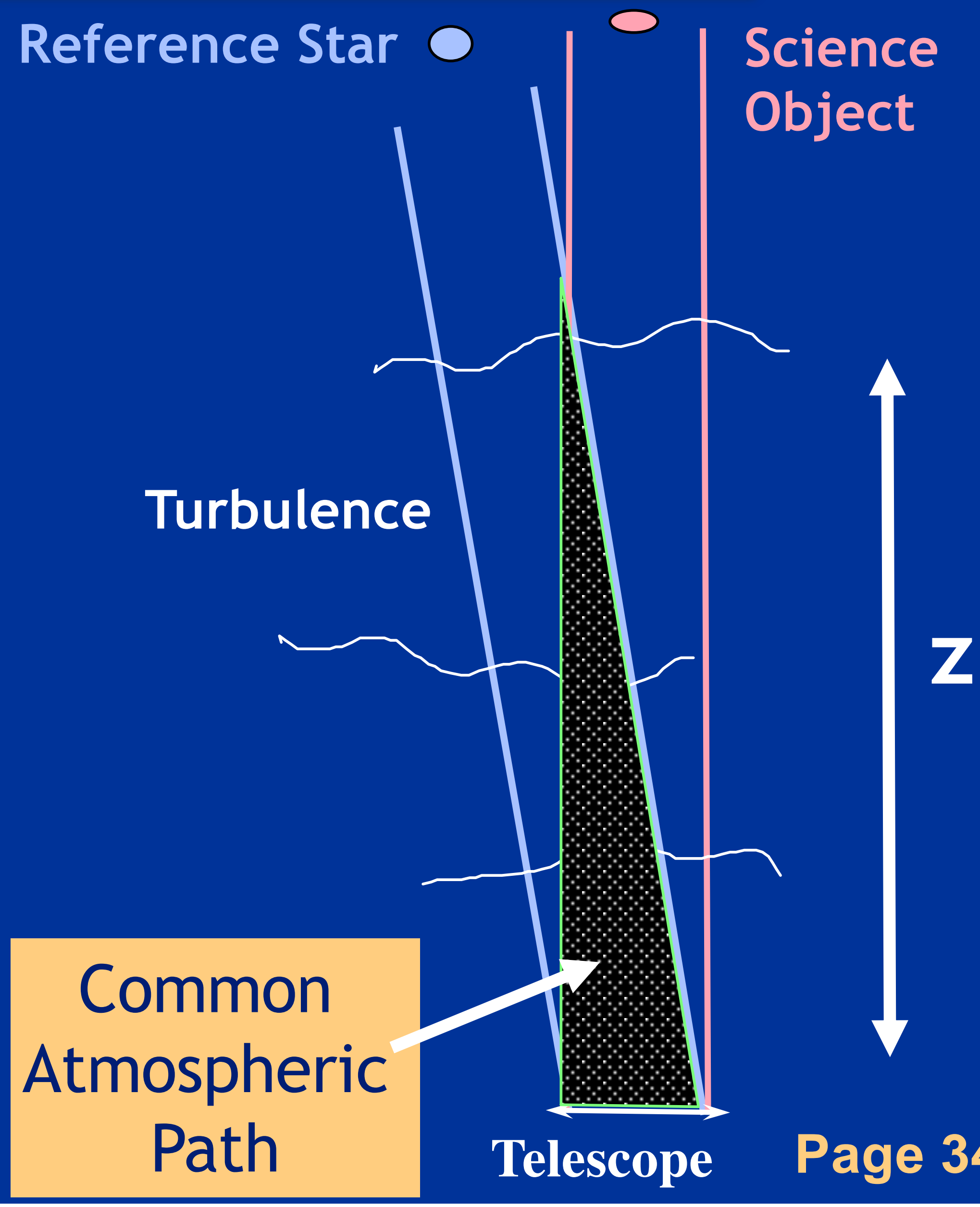
- Composite J, H, K band image, 30 second exposure in each band
  - J band λ = 1.2 μm, H band λ = 1.6 μm, K band λ = 2.2 μm
- Field of view is 40”x40” (at 0.04 arc sec/pixel)

What determines how close the reference star has to be?

Turbulence has to be similar on path to reference star and to science object

Common path has to be large

Anisoplanatism sets a limit to distance of reference star from the science object



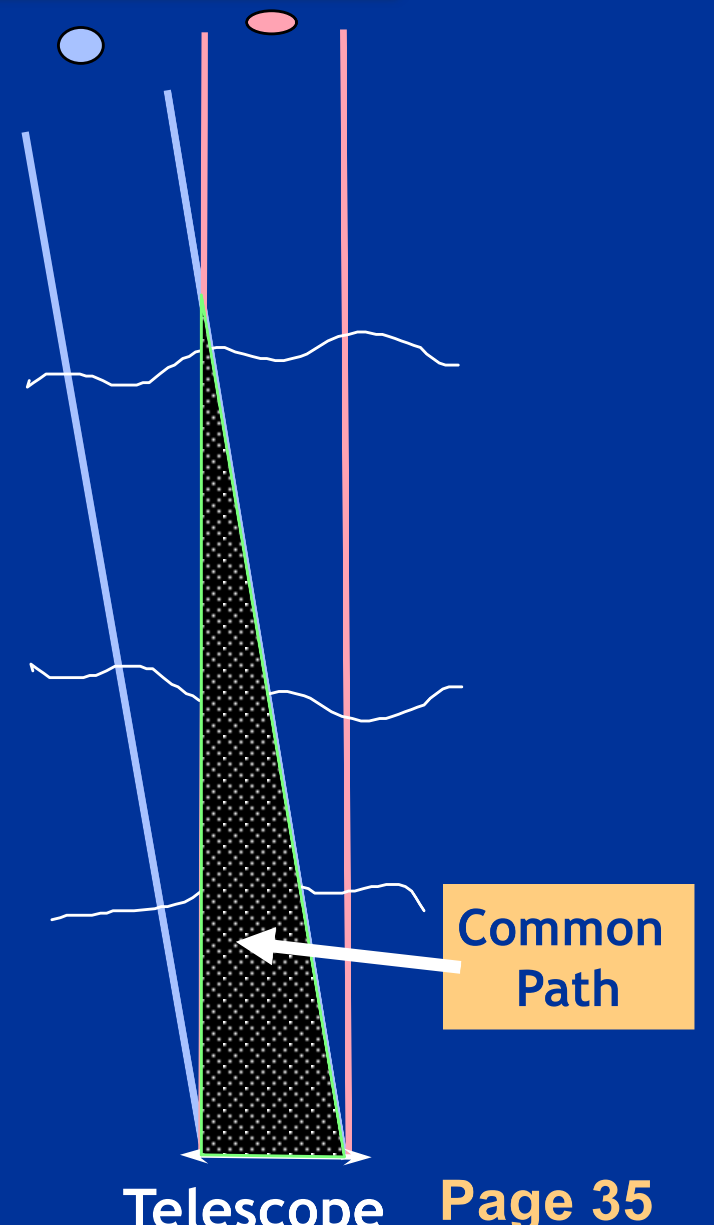
Expression for isoplanatic angle θ_0

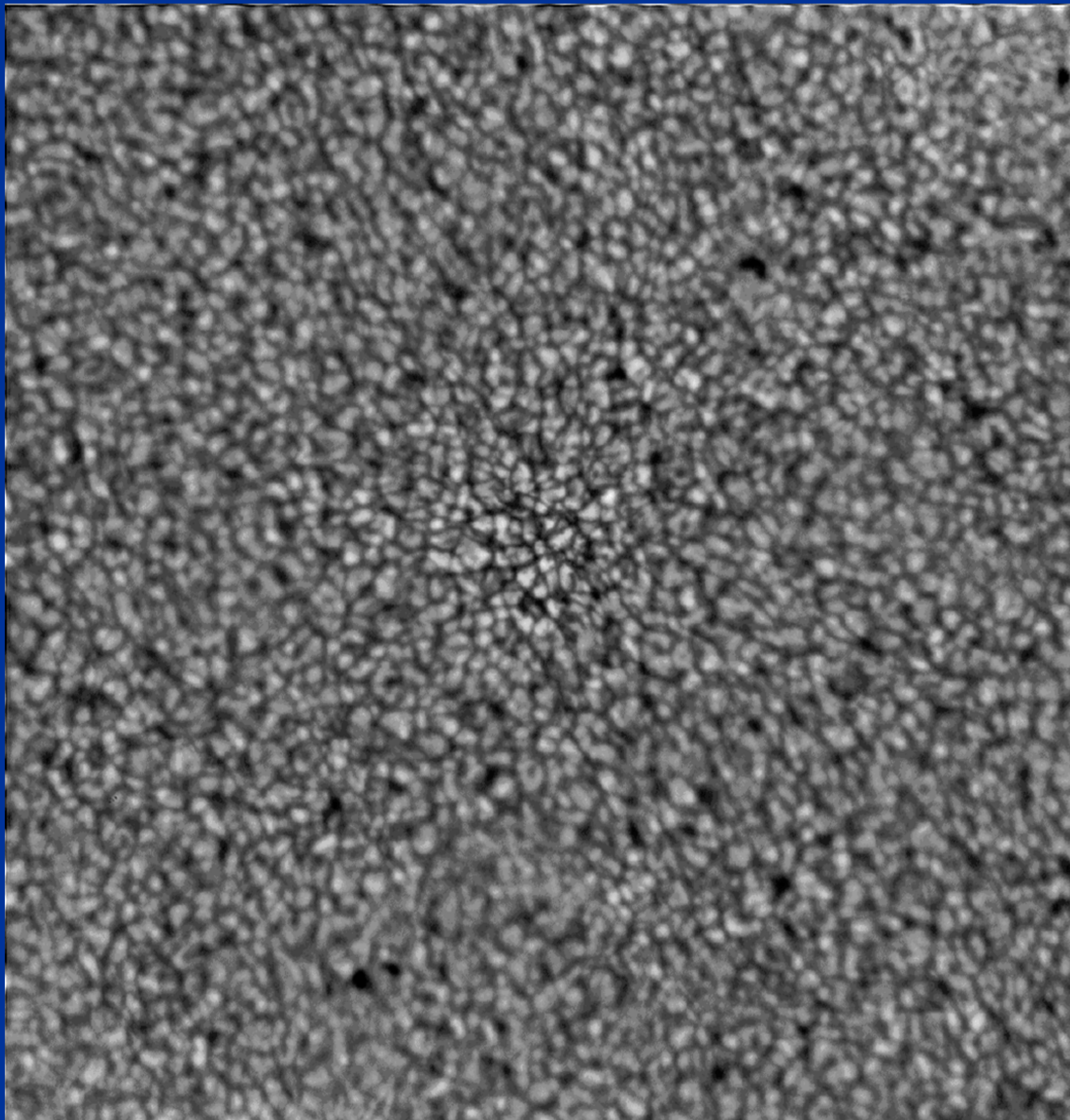
Definition of isoplanatic angle θ_0

$$\frac{Strehl(\theta = \theta_0)}{Strehl(\theta = 0)} = \frac{1}{e} \cong 0.37$$

$$\vartheta_0 = \left[2.914\, k^2 (\sec\zeta)^{8/3} \int_0^\infty dz\, C_N^2(z)\, z^{5/3} \right]^{-3/5}$$

- θ_0 is weighted by <u>high-altitude</u> turbulence ($z^{5/3}$)
- If turbulence is only at low altitude, overlap of the two beams is very high.
- If there is strong turbulence at high altitude, not much turbulence is in common path





More about anisoplanatism:

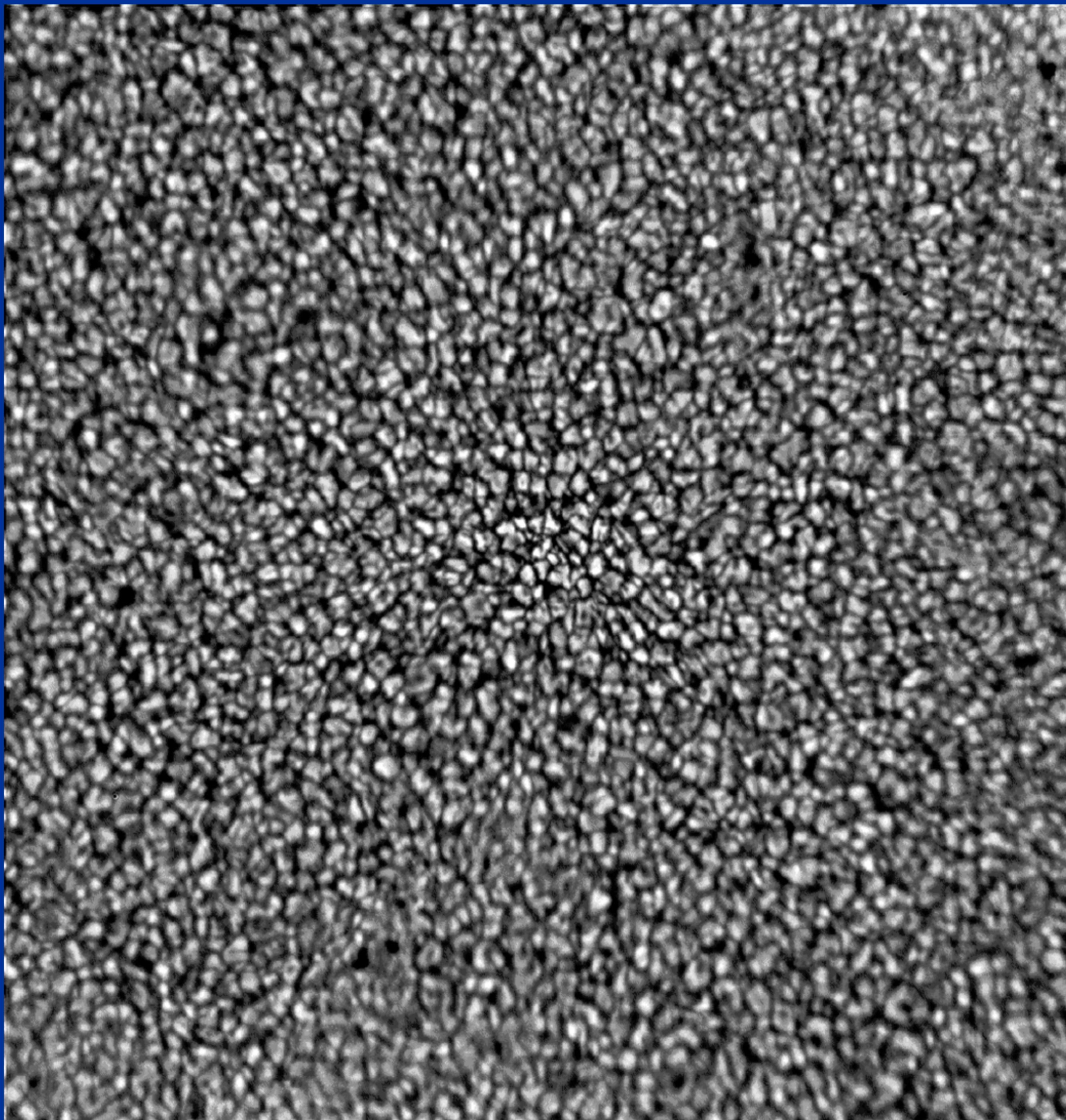
AO image of sun in visible light

11 second exposure

Fair Seeing

Poor high altitude conditions

From T. Rimmele



AO image of sun in visible light:

11 second exposure

Good seeing

Good high altitude conditions

From T. Rimmele

Isoplanatic angle, continued

- Isoplanatic angle θ_0 is weighted by $[\, z^{5/3}\, C_N^2(z)\,]^{3/5}$
- Simpler way to remember θ_0

$$\theta_0 = 0.314\ (\cos\zeta) \left(\frac{r_0}{\bar{h}} \right) \quad \text{where } \bar{h} \equiv \left(\frac{\int dz\ z^{5/3} C_N^2(z)}{\int dz\ C_N^2(z)} \right)^{3/5}$$

Review of atmospheric parameters that are key to AO performance

- r_0 ("Fried parameter")
  - Sets number of degrees of freedom of AO system $N \propto \left(\frac{D}{r_0}\right)^2$
- τ_0 (or Greenwood Frequency ~ $1 / \tau_0$)

$\tau_0 \sim r_0 / V$ where $$V \equiv \left[\frac{\int dz\, C_N^2(z)\, |V(z)|^{5/3}}{\int dz\, C_N^2(z)} \right]^{3/5}$$

  - Sets timescale needed for AO correction, is proportional to r_0
- θ_0 (isoplanatic angle)

$$\theta_0 \cong 0.3 \left(\frac{r_0}{\bar{h}}\right) \quad \text{where } \bar{h} \equiv \left(\frac{\int dz\, C_N^2(z)\, z^{5/3}}{\int dz\, C_N^2(z)} \right)^{3/5}$$

  - Angle for which AO correction applies